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Hyperbolic Systems of Conservation Laws, the Weyl Equation, and Multidimensional Upwinding

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All linear hyperbolic systems of two conservation laws can be transformed to essentially one prototype system. This system can be identified with the Weyl equation of relativistic quantum mechanics. We derive a wave model for this equation and compare the resulting fluctuation splitting scheme with standard dimensional splitting schemes. © 1994 Academic Press, Inc.

1. INTRODUCTION

Upwind schemes have been highly successful for computing solutions to hyperbolic systems of conservation laws in one space dimension. They decompose the solution locally into a sum of finitely many simple waves, each of which can only travel to the left or right. Unfortunately, there is no straightforward generalization of this procedure to multidimensional systems, since the "wind" can now come from infinitely many directions.

In the important article [13], Roe developed the idea of a wave model for the Euler equations of gas dynamics. This ansatz chooses only finitely many directions from multidimensional data and approximates the solution locally by a sum of simple waves traveling into those directions. Philosophically, this is very appealing, since these directions depend only on the local gradient of the solution and not directly on the grid directions.

Fluctuation splitting schemes transport a scalar wave over a multidimensional grid. Using a wave model, they can be applied to multidimensional systems (see [3, 4, 15] and the references therein).

In this article we consider a prototype system of two conservation laws in two space dimensions [9, 5] which we can identify with the Weyl equation of relativistic quantum mechanics. This equation governs the propagation of a particle with spin $\frac{1}{2}$ and mass zero. The main result of this article is the derivation of a wave model for the Weyl equation. We present first numerical experiments comparing the fluctuation splitting scheme based on our wavemodel to standard dimensional splitting schemes.

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2. WAVE MODELS

Let us briefly summarize some ideas outlined in [13, 16]. Let $x=(x_1, ..., x_d) \in \mathbb{R}^d$, $A=(A_1, ..., A_d)$, where the A_j are constant $n \times n$ matrices such that for all unit vectors $\xi = (\xi_1, ..., \xi_d) \in S^{d-1} \subset \mathbb{R}^d$ the matrix $\xi \cdot A = \sum_{j=1}^n \xi_j A_j$ has real eigenvalues $\lambda_m(\xi)$, $m=1, ..., n$, and a complete set of eigenvectors $\{p_m(\xi) \mid m=1, ..., n\}$. Consider the initial value problem

$$\frac{\partial}{\partial t} u(x,t) + \sum_{j=1}^{d} A_j \frac{\partial}{\partial x_j} u(x,t) = 0,$$

$$u: \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^n, \tag{2.1}$$

$$u(x,0) = u_0(x). \tag{2.2}$$

The following observation is due to Roe, Struijs, and Deconinck [16].

LEMMA 1. *Let ∇u be a constant $n \times d$ matrix and suppose that*

$$u_0(x) = u_0(0) + x \cdot \nabla u. \tag{2.3}$$

Let $N \geq 1$ be given, and for $k=1, ..., N$ let $m(k) \in \{1, ..., n\}$, $\alpha^k \in \mathbb{R}$, and $\xi^k \in S^{d-1}$. Let $\lambda^k := \lambda_{m(k)}(\xi^k)$ and $p^k := p_{m(k)}(\xi^k)$. If

$$\nabla u = \sum_{k=1}^{N} \alpha^k \xi^k p^k, \tag{2.4}$$

then the solution of the initial-value-problem (2.1)–(2.3) *is given by*

$$u(x,t) = u_0(0) + \sum_{k=1}^{N} \alpha^k (x \cdot \xi^k - \lambda^k t)\, p^k. \tag{2.5}$$

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For the proof just differentiate (2.5) and use the relation $(\xi^k \cdot A)\,p^k = \lambda^k p^k$. The significance of the lemma is that the solution of (2.1)–(2.3) is decomposed into finitely many planar waves. Upwind schemes in one space dimension are built upon such a decomposition (compare [13]). Formula (2.4) is a system of $n \cdot d$ equations for the unknowns $m(k)$, α^k, ξ^k. A wave model now consists of a choice of some of these unknowns such that for any given Jacobi matrix ∇u, the resulting system (2.4) can be solved uniquely for the other unknowns.

For any system, there is always the "trivial" wave model, which consists of prescribing d independent directions ξ^k and using the full set of eigenvectors for each direction. The unknowns are then α^k, $k=1, ..., N=n \cdot d$. Here no directional information is extracted from ∇u. Numerically, this corresponds to computing only in the grid directions. The best situation is when all the matrices A_j commute. In this case they possess a common set of eigenvectors, and we can diagonalize the system. We can now extract one direction ξ^k and one amplitude α^k for each of the n components ($N=n$).

The more interesting cases lie in-between. For the Euler equations of gas dynamics, first wave models have been derived by Roe [13], followed by [11]. We now proceed to derive wave models for arbitrary systems of two linear conservation laws in two space dimensions.

3. SYSTEMS OF TWO CONSERVATION LAWS AND THE WEYL EQUATION

As a class of model problems, let us consider the case $n=2$ and $d \geqslant 1$. According to a theorem of Gilquin, Laurens, and Rosier [5], all those systems can be transformed either to

$$\frac{\partial}{\partial t} u(x, t) = 0$$

or

$$\frac{\partial}{\partial t} u(x, t) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x_1} u(x, t) = 0$$

or

$$\frac{\partial}{\partial t} u(x, t) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x_1} u(x, t) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x_2} u(x, t) = 0. \tag{3.1}$$

The first two cases correspond to systems which can be diagonalized. We are here interested in the third case. First we will show how (3.1) arises in relativistic quantum mechanics. Afterwards, we will derive a wave model for (3.1).

The Pauli-matrices (see [8]) are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $p_0 = (\hbar/i)(\partial/\partial t)$, $p_j = (\hbar/i)(\partial/\partial x_j)$, $j=1, 2, 3$, be the momentum operators. Now Dirac's equation for the four-component wave function of a particle with spin $\frac{1}{2}$ and mass m is

$$\begin{pmatrix} 0 & p_0 + p \cdot \sigma \\ p_0 - p \cdot \sigma & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = m \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

For a particle with mass zero[1], Dirac's equation reduces to the Weyl equation

$$\begin{aligned} (p_0 + p \cdot \sigma)\eta &= 0 \\ (p_0 - p \cdot \sigma)\xi &= 0. \end{aligned} \tag{3.2}$$

If solutions to (3.2) are independent of x_2, the Weyl equation is equivalent to (3.1). From now on we will denote the independent variables by $(x, y) \in \mathbb{R}^2$ and the dependent variables by $(u, v) \in \mathbb{R}^2$.

In order to derive a wave model we rewrite Weyl's equation in complex form: Let $U = u + iv$, $z = x + iy$, $\bar{U} = u - iv$, and $(\partial/\partial z) = \frac{1}{2}(\partial/\partial x - i(\partial/\partial y))$. Then (3.1) is equivalent to

$$\frac{\partial}{\partial t} \bar{U}(z, \bar{z}, t) + 2 \frac{\partial}{\partial z} U(z, \bar{z}, t) = 0.$$

Let $\hat{U}_z = \frac{1}{2}\mu e^{i\sigma}$ and $\hat{U}_{\bar{z}} = \frac{1}{2} v e^{i\varphi}$ be complex constants, and suppose that the initial data are linear and given by

$$U(z, \bar{z}, 0) = z\hat{U}_z + \bar{z}\hat{U}_{\bar{z}}.$$

We now make the ansatz

$$\begin{aligned} \xi^k &= (\cos\theta^k, \sin\theta^k), \\ \theta^k &= \theta_0 + \frac{2\pi}{3}(k-2), \qquad k = 1, ..., 3. \end{aligned} \tag{3.3}$$

The analogue of (2.3) is

$$U(z, \bar{z}, 0) = \sum_{k=1}^{3} \alpha^k e^{i\theta^k/2} \zeta^k,$$

where

$$\zeta^k = \tfrac{1}{2}(z e^{-i\theta^k} + \bar{z} e^{i\theta^k}).$$

[1] It is still controversial whether the neutrino has zero mass.

Therefore,

$$\mu e^{i\sigma} = \sum_{k=1}^{3} \alpha^k e^{-i\theta^k/2} \tag{3.4}$$

$$\nu e^{i\varphi} = \sum_{k=1}^{3} \alpha^k e^{3i\theta^k/2}. \tag{3.5}$$

From (3.5) and (3.3),

$$\nu e^{i(\varphi - 3\theta_0/2)} = \sum_{k=1}^{3} \alpha^k (-1)^k,$$

so

$$\theta_0 = 2\varphi/3$$

and

$$\alpha^2 = \nu + \alpha^1 + \alpha^3. \tag{3.6}$$

Now (3.4) and (3.6) can be solved easily: for $\psi := \sigma + \theta_0/2$, we obtain

$$\alpha^k = \frac{(-1)^k}{3}\nu + \frac{2}{3}\mu \cos\left(\psi + (k-2)\frac{\pi}{2}\right), \qquad k = 1, 2, 3.$$

4. A NUMERICAL EXPERIMENT

As a first numerical experiment, we compute a space-periodic train of discontinuities traveling obliquely over a two-dimensional cartesian grid. We compare the following three schemes:

(a) The fluctuation splitting *NN*-scheme [15, 16], which we can apply to the Weyl equation using our wave model. In order to obtain a triangular grid, we divide each cell of the cartesian grid along its subdiagonal.

(b) The one-dimensional upwind scheme, which we apply to the two-dimensional problem via alternate dimensional splitting. The one-dimensional scheme is monotone and hence convergent. For scalar multidimensional problems, the splitting scheme converges due to a result of Crandall and Majda [2].

(c) The one-dimensional piecewise linear method with Roe's superbee flux-limiter [12] and Strang's dimensional splitting [18]. The one-dimensional scheme is formally second-order accurate in both space and time and total-variation-diminishing, hence convergent for one-dimensional scalar conservation laws [7].

The initial data are given as follows: Let $\varphi \in [0, \pi/2]$ be a fixed angle and suppose that the solution $u(x, y, t)$, $v(x, y, t)$ of (3.1) depends only on $\xi := x\cos\varphi + y\sin\varphi$ and t. Then

$$u(x, y, t) = w(\xi, t)\cos(\varphi/2) + z(\xi, t)\sin(\varphi/2)$$
$$v(x, y, t) = w(\xi, t)\sin(\varphi/2) - z(\xi/t)\cos(\varphi/2),$$

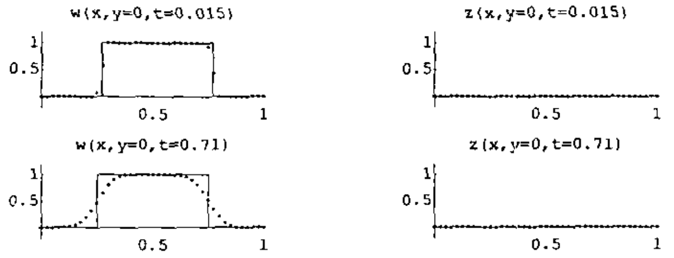

FIG. 4.1. Fluctuation splitting scheme, diagonal flow, $\gamma \approx 0.87$, cross section of the computed solution w_h, z_h versus the exact solution at times $t = \Delta t$ and $t = T$.

where w and z satisfy

$$\frac{\partial}{\partial t} w(\xi, t) + \frac{\partial}{\partial \xi} w(\xi, t) = 0, \qquad \frac{\partial}{\partial t} z(\xi, t) - \frac{\partial}{\partial \xi} z(\xi, t) = 0.$$

We suppose that w and z are space-periodic with period $p = \cos\varphi$ and assign initial data

$$w_0(\xi) := \begin{cases} 1, & p/4 \leqslant \xi < 3p/4 \\ 0, & 3p/4 \leqslant \xi < 5p/4 \end{cases}$$

$$z_0(\xi) \equiv 0,$$

a single wave traveling to the right with speed $a = 1$.

Let $h = \Delta x = \Delta y$ be the mesh size of the cartesian grid, and let $\Delta\xi := h/(\cos\varphi + \sin\varphi)$. If we define the CFL-like number $\gamma := a\,\Delta t/\Delta\xi$, then the fluctuation splitting scheme would be stable for the scalar equation

$$\frac{\partial}{\partial t} u(x, y, t) + a(\cos\varphi, \sin\varphi) \cdot \nabla u(x, y, t) = 0,$$

provided that $\gamma \leqslant 1$. Let now $T = p/a$ be the time period and let $M = T/\Delta t$ be the number of time-steps per period. Then $\gamma = aT/M\,\Delta\xi$.

4.1. *Diagonal Flow*, $\varphi = \pi/4$

In our first calculation, we used $\varphi = \pi/4$ and $M = 46$, corresponding to $\gamma \approx 0.87$. In Figs. 4.1–4.3, we display cross

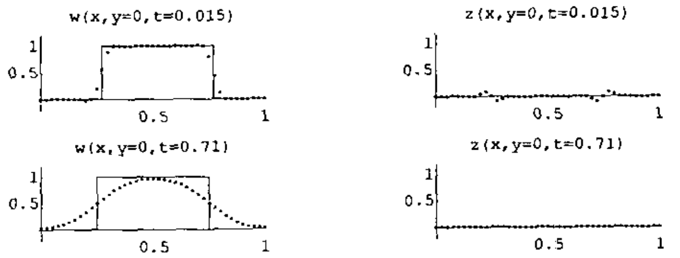

FIG. 4.2. First-order upwind scheme. Parameters as in Fig. 4.1. Note the oscillations in z_h after one timestep.

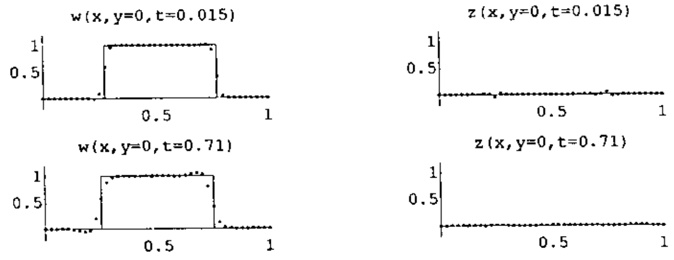

FIG. 4.3. Piecewise linear method with superbee limiter. Parameters as in Fig. 4.1. The oscillations persist.

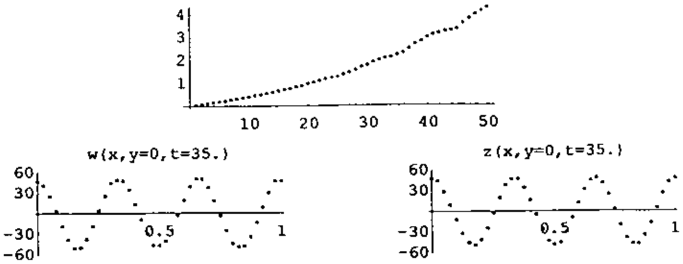

FIG. 4.4. Piecewise linear method. Diagonal flow, $\gamma \approx 0.87$: (a) $\log(\|(u_h, v_h)\|_{L_\infty})$ versus t/T; 50 time-periods. (b) Cross section of the solution at $t = 50T$. The figures demonstrate the exponential growth of the oscillations.

sections of the solution for the three schemes at times $t = \Delta t$ and $t = T$, where $T = p/a$ is the time period. The fluctuation splitting scheme shows a result typical for a first-order scheme (see Fig. 4.1). The dimensional splitting schemes, however, introduce oscillations. Such oscillations were already observed and analyzed in [14, 10]: when solving the one-dimensional Riemann problem in the x- and y-directions, these schemes do not recognize the *single* wave traveling obliquely to the grid. Insteady, they solve the Riemann problem by a superposition of *two* waves. Thus they introduce an artificial intermediate state which causes the oscillations seen in Figs. 4.2 and 4.3. For the first-order upwind scheme, the oscillations are damped and disappear quickly. For the high-resolution piecewise linear method, however, they persist and grow exponentially in time: in Fig. 4.4 we show $\|(u_h(t), v_h(t))\|_{L_\infty}$ in logarithmic scale. This exponential growth happens even when reducing the timestep by a factor of 10 (see Fig. 4.5).

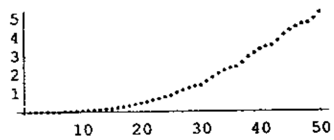

FIG. 4.5. Same as Fig. 4.4(a), but with CFL number $\gamma \approx 0.087$.

4.2. *Oblique Flow*, $\varphi = \arctan(\frac{1}{2})$

One might argue that the particular direction of flow chosen in the previous test calculation gives the fluctuation splitting scheme an unfair advantage over the dimensional splitting schemes, since diagonal flow is aligned with the triangular grid used by that scheme. We thus present the analogous calculation for oblique flow, $\varphi = \arctan(\frac{1}{2})$, $M = 54$ (i.e., $\gamma \approx 0.89$). In Fig. 4.6, we show the solution of all three schemes at $t = T$ and in Fig. 4.7 the exponential growth of the L_∞ norm for the piecewise linear method. The results are not as pronounced as for the diagonal flow, but qualitatively similar.

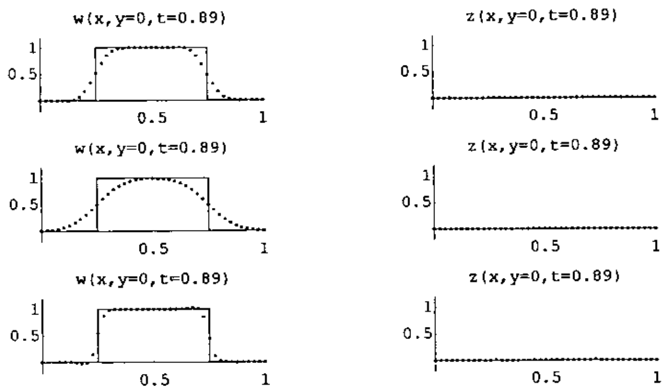

FIG. 4.6. Fluctuation splitting, first-order upwind, and piecewise linear schemes, oblique flow, $\gamma \approx 0.89$, cross section of the computed solutions w_h, z_h versus the exact solution at time $t = T$.

In order to demonstrate the ability of the fluctuation splitting scheme to recognize the relevant direction of wave propagation, we display the *fluctuation transport vectors* in Fig. 4.8 at time $t = T$. These vectors, which are attached to the barycenter of each triangle, are defined by

$$|\alpha^k| \, \xi^k = |\alpha^k| (\cos \theta^k, \sin \theta^k), \qquad k = 1, 2, 3.$$

In most triangles, only one of these vectors is visible, and for this vector, θ^k is close to φ. This shows that most of the

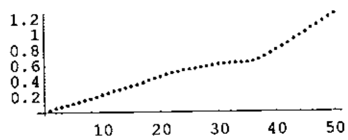

FIG. 4.7. Piecewise linear method, oblique flow, $\gamma \approx 0.89$, $\log(\|u_h, v_h)\|_{L_\infty})$ versus t/T. Exponential growth of the oscillations.

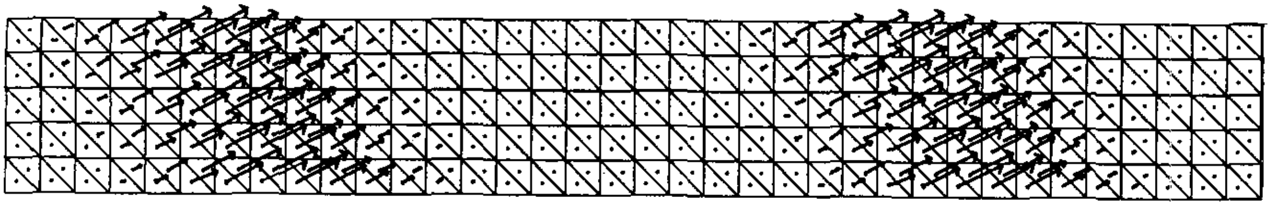

FIG. 4.8. Fluctuation transport vectors. Oblique flow, $\gamma \approx 0.89$, $t = T$.

gradient is projected onto the correct eigenvector. Finally, we would like to remark that reversing the diagonals of the grid shown in Fig. 4.8 makes the fluctuation splitting scheme somewhat less stable.

4.3. *Further Test Problems*

The Weyl equation is a particular example of the following class of nonlinear hyperbolic conservation laws, which was proposed by Lax [9],

$$\frac{\partial}{\partial t}\bar{U}(z, \bar{z}, t) + \frac{\partial}{\partial z} F(U(z, \bar{z}, t)) = 0,$$

where $F(U)$ is holomorphic and we have used the notation of Section 3. In particular, let $F(U) := (2/(n+1))\, U^{n+1}$. For $n = 0$, we obtain the Weyl equation and for $n = 1$ the complex Burgers equation. This equation possesses interesting solutions to the one-dimensional Riemann problem, including nonstandard "crossing" shocks [17, 6, 1]. For all $n \geq 0$, there exist radially symmetric solutions which are, however, multiple-valued. Linearizing the systems over a triangle and using the wave model locally, the fluctuation splitting schemes may be applied to these nonlinear problems.

5. CONCLUSION

We have presented a prototype linear 2×2 system of conservation laws in two space dimensions, identified it with the Weyl equation and derived a nontrivial multidimensional wave model. We computed discontinuities traveling obliquely over a rectangular grid. A state-of-the-art high-resolution dimensional splitting scheme produces unacceptably large oscillations, which grow exponentially in time. The fluctuation splitting scheme based on our wave model recognizes the direction of flow well, but it is only first-order accurate for time-dependent flow (such schemes may achieve second-order resolution for steady state calculations [4]). The development of genuinely multidimensional second-order resolution methods for systems of conservation laws remains a challenging task. The Weyl equation, being the simplest generalization of the one-dimensional scalar advection equation to multidimensional systems, provides simple, yet interesting, test problems for these future schemes.

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